

# Quantum Wave Equation of Photon

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**Abstract** In this paper, we give the quantum wave equations of single photon when it is in the vacuum and medium. With these wave equations, we can study light interference and diffraction with the approach of quantum theory, and also can study the quantum property of photon when it is in a general crystal and photonic crystal. Otherwise, it can be applied in quantum optics and condensed matter field.

**Keywords** Photon wave function · Quantum wave equation · Photonic crystal

## 1 Introduction

It is known that one can describe single-photon states using a photon-as-particle viewpoint, specifying the photon wave function. The photon wave function and its equation of motion are established from the Einstein energy-momentum-mass relation, assuming a local energy density. According to modern quantum field theory, photons, together with all other particles, are the quantum excitations of a field. In the case of photons, these are the excitations of the electromagnetic field. The lowest field excitation of a given type corresponds to one photon and higher field excitations involve more than one photon. This concept of a photon enables one to use the photon wave function not only to describe quantum states of an excitation of the free field but also of the electromagnetic field interacting with a medium.

The photon wave function can not have all the properties of the Schrodinger wave function of nonrelativistic wave mechanics. Insistence on those properties that, owing to peculiarities of photon dynamics, cannot be rendered, led some physicists to the extreme opinion that the position probability density photon wave function does not exist, and the photon wave function is energy-density wave function in coordinate space, which has developed over the past dozen years [1–4]. We know that the quantized field theory of light developed by Dirac [5], and this actually provides a derivation of the Maxwell equations, starting from

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fundamental principles. The derivation parallels that of Dirac for the electron and its quantum field [6, 7]. A key difference between the electron and photon derivations has to do with the famous localization problem for the photon [8]. Whereas non-relativistic electrons can be in a position eigenstate, at least in principle, a photon cannot. On the other hand, the energy density of the electromagnetic field in free space can be expressed as a local quantity,  $E^2(x) + c^2 B^2(x)$ . Since the photon wave function is the sum of real and imaginary parts, i.e.,  $\vec{\psi}(\vec{r}, t) = 2^{-\frac{1}{2}}(\vec{E}(\vec{r}, t) + i\vec{H}(\vec{r}, t))$ . We call the mean-energy density wave function or the Bialynicki-Birula-Sipe wave function and its equation of motion is the photon wave equation. This section introduces the photon wave function and its meaning. Section 2 gives the potential energy of photon interaction with medium firstly and give the energy of photon in medium. Section 3 gives the quantum wave equation of single photon in the medium by extending the method as Ref. [9]. Section 4 gives the application of quantum wave equation of photon.

## 2 Relativistic Hamiltonian for a Photon

We know a relativistic Lagrangian for a particle in external field is

$$L = -m_0c^2\sqrt{1 - \frac{v^2}{c^2}} - V, \tag{1}$$

where  $m_0$  is the rest mass of the particle,  $v$  is the velocity of the particle,  $c$  is the velocity of light and  $V$  is the potential energy of the particle in external field. The canonical momentum  $\vec{p}$  conjugate to the position coordinate  $\vec{x}$  is obtained by the definition,

$$p_i = \frac{\partial L}{\partial v_i} = \frac{m_0v_i}{\sqrt{1 - \frac{v^2}{c^2}}}, \tag{2}$$

and so

$$\vec{P} = \frac{m_0\vec{v}}{\sqrt{1 - \frac{v^2}{c^2}}}, \tag{3}$$

the Hamiltonian takes on the form:

$$\begin{aligned} H = E = \vec{P} \cdot \vec{v} - L &= \frac{m_0c^2}{\sqrt{1 - \frac{v^2}{c^2}}} + V \\ &= \sqrt{c^2p^2 + m_0^2c^4} + V, \end{aligned} \tag{4}$$

where  $E$  is the total energy of the particle.

For a photon, the rest mass is  $m_0 = 0$ , and the (4) becomes

$$E = cp + V. \tag{5}$$

In medium, the energy, momentum and velocity of photon are

$$E = h\nu, \quad p = \frac{h}{\lambda}, \quad v = \nu\lambda, \tag{6}$$

where  $\nu$  is photon frequency, and  $\lambda$  is photon wavelength. Substituting (6) into (5), we can obtain

$$h\nu = hc + V\lambda. \quad (7)$$

From (7), we give the potential energy of photon in a medium

$$V = \frac{h(\nu - c)}{\lambda} = \frac{h\left(\frac{c}{n} - c\right)}{\lambda} = \frac{hc}{\lambda} \left(\frac{1}{n} - 1\right). \quad (8)$$

Substituting (8) into (5), we can give the energy of photon in medium

$$E = cp + V = p\nu. \quad (9)$$

### 3 Quantum Wave Equation of Photon in Medium

With the photon energy equation (5), we can study the quantum wave equation of photon in medium. Equation (5) can be written as

$$E = c\sqrt{\vec{p} \cdot \vec{p}} + V. \quad (10)$$

Define a multicomponent wave function  $\vec{\psi}(\vec{p}, E)$  obeying the normalization condition

$$(2\pi\hbar)^{-3} \int d^3p \vec{\psi}^*(\vec{p}, E) \cdot \vec{\psi}(\vec{p}, E) = 1. \quad (11)$$

Since a photon spin  $s = 1$ , its wave function has three components, i.e.,  $\vec{\psi}(\vec{p}, E) = (\psi_x(\vec{p}, E), \psi_y(\vec{p}, E), \psi_z(\vec{p}, E))$ , we multiply (10) by the wave function  $\vec{\psi}(\vec{p}, E)$  to give

$$E\vec{\psi}(\vec{p}, E) = (c\sqrt{\vec{p} \cdot \vec{p}} + V)\psi(\vec{p}, E). \quad (12)$$

In order to represent the square-root operator  $\sqrt{\vec{p} \cdot \vec{p}}$ , we define a vector operator  $\hat{A} = i\vec{p} \times$ , where  $\times$  is the cross product operator [9], we have

$$\hat{A}\hat{A}\vec{\psi} = -\vec{p} \times (\vec{p} \times \vec{\psi}) = (\vec{p} \cdot \vec{p})\vec{\psi} - \vec{p}(\vec{p} \cdot \vec{\psi}). \quad (13)$$

Any vector field can be written as the sum of two linearly independent part,  $\vec{\psi} = \vec{\psi}_T + \vec{\psi}_L$ , where the transverse part  $\vec{\psi}_T$  and longitudinal part  $\vec{\psi}_L$  obey

$$\vec{p} \cdot \vec{\psi}_T = 0, \quad \vec{p} \times \vec{\psi}_L = 0, \quad (14)$$

and (13) becomes

$$\hat{A}\hat{A}\vec{\psi} = \vec{p} \cdot \vec{p}(\vec{\psi}_T + \vec{\psi}_L) - \vec{p}(\vec{p} \cdot (\vec{\psi}_T + \vec{\psi}_L)). \quad (15)$$

Identifying the transverse part  $\vec{\psi}_L$  as the relevant field for the photon. For the photon field  $\vec{\psi}_L$ , (15) becomes

$$\hat{A}\hat{A}\vec{\psi}_T = \vec{p} \cdot \vec{p}\vec{\psi}_T, \quad (16)$$

and then

$$\sqrt{\hat{A}\hat{A}}\vec{\psi}_T = \hat{A}\vec{\psi}_T = \sqrt{\vec{p} \cdot \vec{p}}\vec{\psi}_T, \quad (17)$$

we multiply (17) by the light velocity  $c$  and substitute (12) into (17) to give

$$c\hat{A}\vec{\psi}_T = c\sqrt{\vec{p} \cdot \vec{p}}\vec{\psi}_T = (E - V)\vec{\psi}_T, \tag{18}$$

or

$$(E - V)\vec{\psi}_T(\vec{p}, E) = ci\vec{p} \times \vec{\psi}_T(\vec{p}, E). \tag{19}$$

From (19), we know  $\vec{\psi}_T(\vec{p}, E)$  must be a complex-valued vector, we can make Fourier transform for wave function  $\vec{\psi}(\vec{p}, E)$  from momentum space to coordinate space, and from energy to time, accounting for the constraint between energy and momentum ( $E = c|\vec{p}|$ ) by including a delta function. The energy  $E$  to be considered as an independent variable, and gives [9]

$$\vec{\psi}(\vec{r}, t) = (2\pi\hbar)^{-4} \int \int dE d^3 p \delta(E - v|\vec{p}|) \exp(-iEt/\hbar + i\vec{p} \cdot \vec{r}/\hbar) f(E)\vec{\psi}(\vec{p}, E). \tag{20}$$

The weight function  $f(E)$  has been included to allow different forms of normalization of the coordinate-space function  $\vec{\psi}_T(\vec{r}, t)$ . For the photon, we adopt the choice in Ref. [9], which gives for the coordinate-space normalization,

$$\int d^3 r \vec{\psi}^*(\vec{r}, t) \cdot \vec{\psi}(\vec{r}, t) = (2\pi\hbar)^{-3} \int d^3 p E(p) \vec{\psi}^*(\vec{p}, E) \cdot \vec{\psi}(\vec{p}, E) = \langle E \rangle, \tag{21}$$

where the  $\langle E \rangle$  denotes the expectation value of the photon’s energy. For the photon wave function  $\vec{\psi}_T(\vec{r}, t)$ , there is the same transform as  $\vec{\psi}(\vec{r}, t)$

$$\vec{\psi}_T(\vec{r}, t) = (2\pi\hbar)^{-4} \int \int dE d^3 p f(E) \delta(E - v|\vec{p}|) \exp(-iEt/\hbar + i\vec{p} \cdot \vec{r}/\hbar) \vec{\psi}_T(\vec{p}, E). \tag{22}$$

From (22), there are the reversal Fourier transforms as follows:

$$\begin{aligned} & E \cdot f(E) \delta(E - v|\vec{p}|) \vec{\psi}_T(\vec{p}, E) \\ &= (2\pi\hbar)^{-4} \int \int dt d^3 r E \cdot \exp(iEt/\hbar - i\vec{p} \cdot \vec{r}/\hbar) \vec{\psi}_T(\vec{r}, t) \\ &= (2\pi\hbar)^{-4} \int \int dt d^3 r \frac{\hbar}{i} \frac{\partial}{\partial t} \{ \exp[iEt/\hbar - i\vec{p} \cdot \vec{r}/\hbar] \} \vec{\psi}_T(\vec{r}, t) \\ &= (2\pi\hbar)^{-4} \frac{\hbar}{i} \left\{ \int \int dt d^3 r \left( \frac{\partial}{\partial t} [\exp(iEt/\hbar - i\vec{p} \cdot \vec{r}/\hbar) \cdot \vec{\psi}_T(\vec{r}, t)] \right. \right. \\ &\quad \left. \left. - \exp(iEt/\hbar - i\vec{p} \cdot \vec{r}/\hbar) \frac{\partial}{\partial t} \vec{\psi}_T(\vec{r}, t) \right) \right\} \\ &= (2\pi\hbar)^{-4} \frac{\hbar}{i} \left\{ \int \int d^3 r d[\exp(iEt/\hbar - i\vec{p} \cdot \vec{r}/\hbar) \cdot \vec{\psi}_T(\vec{r}, t)] \right. \\ &\quad \left. - \int \int dt d^3 r \exp(iEt/\hbar - i\vec{p} \cdot \vec{r}/\hbar) \frac{\partial}{\partial t} \vec{\psi}_T(\vec{r}, t) \right\} \\ &= (2\pi\hbar)^{-4} i\hbar \int \int dt d^3 r \exp(iEt/\hbar - i\vec{p} \cdot \vec{r}/\hbar) \frac{\partial}{\partial t} \vec{\psi}_T(\vec{r}, t), \tag{23} \end{aligned}$$

$$\begin{aligned}
& \delta(E - v|\vec{p}|)f(E)\vec{p} \times \vec{\psi}_T(\vec{p}, E) \\
&= (2\pi\hbar)^{-4} \int \int dt d^3r \exp(iEt/\hbar - i\vec{p} \cdot \vec{r}/\hbar) \vec{p} \times \vec{\psi}_T(\vec{r}, t) \\
&= (2\pi\hbar)^{-4} \int \int dt d^3r \nabla [\exp(iEt/\hbar - i\vec{p} \cdot \vec{r}/\hbar)] \left(-\frac{\hbar}{i}\right) \times \vec{\psi}_T(\vec{r}, t) \\
&= -(2\pi\hbar)^{-4} \frac{\hbar}{i} \int \int dt d^3r (\nabla [\exp(iEt/\hbar - i\vec{p} \cdot \vec{r}/\hbar)] \times \vec{\psi}_T(\vec{r}, t)) \\
&\quad - \exp(iEt/\hbar - i\vec{p} \cdot \vec{r}/\hbar) \nabla \times \vec{\psi}_T(\vec{r}, t) \\
&= -(2\pi\hbar)^{-4} i\hbar \int \int dt d^3r \exp(iEt/\hbar - i\vec{p} \cdot \vec{r}/\hbar) \nabla \times \vec{\psi}_T(\vec{r}, t), \tag{24}
\end{aligned}$$

and

$$\begin{aligned}
& \delta(E - v|\vec{p}|)f(E)V\vec{\psi}_T(\vec{p}, E) \\
&= (2\pi\hbar)^{-4} \int \int dt d^3r \exp(iEt/\hbar - i\vec{p} \cdot \vec{r}/\hbar) V\vec{\psi}_T(\vec{r}, t) \\
&= (2\pi\hbar)^{-4} i\hbar \int \int dt d^3r \exp(iEt/\hbar - i\vec{p} \cdot \vec{r}/\hbar) \frac{V}{i\hbar} \vec{\psi}_T(\vec{r}, t), \tag{25}
\end{aligned}$$

substituting (23), (24) and (25) into (19), we have

$$i\hbar \frac{\partial}{\partial t} \vec{\psi}_T(\vec{r}, t) = c\hbar \nabla \times \vec{\psi}_T(\vec{r}, t) + V\vec{\psi}_T(\vec{r}, t). \tag{26}$$

Equation (26) is quantum wave equation of photon in medium.

For the free photon ( $V = 0$ ), we have

$$i\hbar \frac{\partial}{\partial t} \vec{\psi}_T(\vec{r}, t) = c\hbar \nabla \times \vec{\psi}_T(\vec{r}, t). \tag{27}$$

Equation (27) is quantum wave equation of free photon, and it is the same as Ref. [9]. Multiplying (27) by  $\hbar$ , we obtain

$$i\hbar \frac{\partial}{\partial t} \vec{\psi}_T(\vec{r}, t) = c\hbar \nabla \times \vec{\psi}_T(\vec{r}, t) = c(\hat{p} \cdot \vec{s})\vec{\psi}_T(\vec{r}, t), \tag{28}$$

with

$$s_x = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad s_y = \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{pmatrix}, \quad s_z = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{29}$$

we find

$$s^2 = s_x^2 + s_y^2 + s_z^2 = 2 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = 2I = s(s+1)I. \tag{30}$$

From (30), we have  $s = 1$ . So (27) and (28) are the particle wave equation, corresponding to spin  $s = 1$  and rest mass  $m_0 = 0$ , i.e., (27) and (28) are the quantum wave equation of photon.

### 4 The Application of the Photon Quantum Wave Equation

In the following, we give some examples about the application of the photon quantum wave equation, which include to study the quantum property of light in general medium and in photonic crystal. We think the equation can be applied in many fields.

If we suppose the medium there are  $N$  molecules per unit volume with  $z$  electrons per molecule, and that, instead of a single binding frequency for all, there are  $f_s$  electrons per molecule with binding frequency  $\omega_s$  and damping constant  $\gamma_s$ , then the medium index  $n$  is given by

$$n^2 = 1 + \frac{Ne^2}{\epsilon_0 m} \sum_s \frac{f_s}{\omega_s^2 - \omega^2 - i\omega\gamma_s}, \tag{31}$$

where  $\omega$  is the frequency of incident photon,  $m$  is the electron mass. Substituting (31) into (8), we can obtain the potential energy of photon in medium

$$V = \frac{hc}{\lambda} \left( \frac{1}{n} - 1 \right) = \frac{hc}{\lambda} \left[ \left( 1 + \frac{Ne^2}{\epsilon_0 m} \sum_s \frac{f_s}{\omega_s^2 - \omega^2 - i\gamma_s\omega} \right)^{-\frac{1}{2}} - 1 \right]. \tag{32}$$

Substituting (32) into (26), we can obtain the quantum wave equation of photon in medium

$$i\hbar \frac{\partial}{\partial t} \vec{\psi}_T(\vec{r}, t) = c\hbar \nabla \times \vec{\psi}_T(\vec{r}, t) + \frac{hc}{\lambda} \left[ \left( 1 + \frac{Ne^2}{\epsilon_0 m} \sum_s \frac{f_s}{\omega_s^2 - \omega^2 - i\gamma_s\omega} \right)^{-\frac{1}{2}} - 1 \right] \vec{\psi}_T(\vec{r}, t). \tag{33}$$

Equation (33) can be solved by the method of separation of variable. By writing

$$\vec{\psi}(\vec{r}, t) = \vec{\psi}(\vec{r})g(t), \tag{34}$$

substituting (34) into (33), we have

$$g(t) = e^{-\frac{i}{\hbar}Et}, \tag{35}$$

and

$$c\hbar \nabla \times \vec{\psi}_T(\vec{r}) = \hbar\omega \left[ 2 - \left( 1 + \frac{Ne^2}{\epsilon_0 m} \sum_s \frac{f_s}{\omega_s^2 - \omega^2 - i\gamma_s\omega} \right)^{-\frac{1}{2}} \right] \vec{\psi}_T(\vec{r}), \tag{36}$$

simplifying (36), we have

$$\nabla \times \vec{\psi}_T(\vec{r}) = k \left[ 2 - \left( 1 + \frac{Ne^2}{\epsilon_0 m} \sum_s \frac{f_s}{\omega_s^2 - \omega^2 - i\gamma_s\omega} \right)^{-\frac{1}{2}} \right] \vec{\psi}_T(\vec{r}). \tag{37}$$

From (37), we can study the quantum property of light in medium.

When the refractive index of medium is periodical in space, the medium is photonic crystal, and its refractive index can be written as:

$$n(\vec{r}) = n(\vec{r} + m\tau) \quad m = 0, 1, 2, \dots, \tag{38}$$

where  $\tau$  is period. Substituting (38) and (8) into (26), we have

$$i\hbar \frac{\partial}{\partial t} \vec{\psi}_T(\vec{r}, t) = c\hbar \nabla \times \vec{\psi}_T(\vec{r}, t) + \frac{\hbar c}{\lambda} \left( \frac{1}{n(\vec{r})} - 1 \right) \vec{\psi}_T(\vec{r}, t) \quad (39)$$

the (39) is quantum wave equation of photon in photonic crystal. From the equation, we can study the quantum property of light in photonic crystal.

## 5 Conclusion

In conclusion, we firstly give the potential energy of photon interaction with medium, and also give the photon quantum wave equation in the vacuum and medium. With these quantum wave equations of photon, we can study light interference and diffraction, and can study the quantum property of light in general medium and photonic crystal.

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